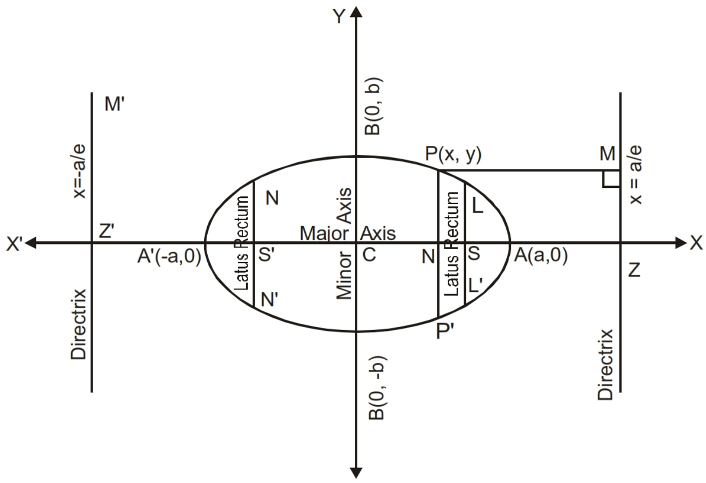
ELLIPSE

EQUATION OF AN ELLIPSE IN STANDARD FORM

The Standard form of the equation of an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b),$$

where a and b are constants.



TERMS RELATED TO AN ELLIPSE

A sketch of the locus of a moving point satisfying the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$, has been shown in the figure given above.

Symmetry

(a) On replacing y by –y, the above equation remains unchanged. So, the curve is symmetrical about x-axis.

(b) On replacing x by –x, the above equation remains unchanged. So, the curve is symmetrical about y-axis

Foci

If S and S' are the two foci of the ellipse and their coordinates are (ae, 0) and (–ae, 0) respectively, then distance between foci is given by

$$SS' = 2ae.$$

Directrices

If ZM and Z' M' are the two directrices of the ellipse and their equations are $x=\frac{a}{e}$ and $x=-\frac{a}{e}$ respectively, then the distance between directrices is given by

$$ZZ' = \frac{2a}{e}.$$

Axes

The lines AA' and BB' are called the major axis and minor axis respectively of the ellipse.

The length of major axis = AA' = 2a

The length of minor axis = BB' = 2b

Centre

The point of intersection C of the axes of the ellipse is called the centre of the ellipse. All chords, passing through C are bisected at C.

Vertices

The end points A and A' of the major axis are known as the vertices of the ellipse

$$A \equiv (a, 0) \text{ and } A' \equiv (-a, 0)$$

Focal chord

A chord of the ellipse passing through its focus is called a focal chord.

Ordinate and Double Ordinate

Let P be a point on the ellipse. From P, draw PN $\perp$ AA' (major axis of the ellipse) and produce PN to meet the ellipse at P'. Then PN is called an ordinate and PNP' is called the double ordinate of the point P.

Latus Rectum

If LL' and NN' are the latus rectum of the ellipse, then these lines are $\perp$ to the major axis AA' passing through the foci S and S' respectively.

$$L \equiv \left(ae, \frac{b^2}{a}\right), \qquad L' \equiv \left(ae, \frac{-b^2}{a}\right)$$

$$N \equiv \left(-ae, \frac{b^2}{a}\right), \qquad N' \equiv \left(-ae, \frac{-b^2}{a}\right)$$

Length of latus rectum = LL' = $\frac{2b^2}{a} = NN'$.

By definition SP = ePM = $e\left(\frac{a}{e} - x\right)$ = a – ex and $S'P = e\left(\frac{a}{e} + x\right) = a + ex$.

Thus implies that distances of any point P(x, y) lying on the ellipse from foci are : (a – ex) and (a + ex). In other words SP + S'P = 2a

i.e., sum of distances of any point P(x, y) lying on the ellipse from foci is constant.

Eccentricity

Since, SP = ePM, therefore

$$SP^2 = e^2\, PM^2$$

or

$$(x - ae)^2 + (y - 0)^2 = e^2\left(\frac{a}{e} - x\right)^2$$

$$(x - ae)^2 + y^2 = (a - ex)^2$$

$$x^2 + a^2e^2 - 2aex + y^2 = a^2 - 2aex + e^2x^2$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

On comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$b^2 = a^2(1 - e^2) \quad \text{or} \quad e = \sqrt{1 - \frac{b^2}{a^2}}$$

Auxillary Circle

The circle drawn on major axis AA' as diameter is known as the Auciliary circle.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then the equation of its auxiliary circle is

$$x^2 + y^2 = a^2.$$

Let Q be a point on auxiliary circle so that QM, perpendicular to major axis meets the ellipse at P. The points P and Q are called as corresponding point on the ellipse and auxiliary circle respectively.

The angle θ is known as eccentric angle of the point P on the ellipse.

It may be noted that the CQ and not CP is inclined at θ with x-axis.

GENERAL EQUATION OF THE ELLIPSE

The general equation of an ellipse whose focus is (h, k) and the directrix is the line ax + by + c = 0 and the eccentricity will be e. Then let $P(x_1, y_1)$ be any point on the ellipse which moves such that

$$SP = CPM$$

$$\Rightarrow \quad (x_1 - h)^2 + (y_1 - k)^2 = \frac{e^2(ax_1+by_1+c)^2}{a^2+b^2}$$

Hence the locus of (x_1, y_1) will be given by

$$(a^2 + b^2)\,[(x - h)^2 + (y - k)^2] = e^2\,(ax + by + c)^2$$

Which is the equation of second degree from which we can say that any equation of second degree represent equation of an ellipse.

Note : Condition for second degree in X and Y to represent an ellipse is thay if $h^2 = ab < 0$ & $\Delta = abc + 2\,fgh - af^2 - by^2 - ch^2 \neq 0$.

PARAMETRIC EQUATION OF THE ELLIPSE

The coordinates x = a cos θ and y = b sin θ satisfy the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

for all real values of θ. Thus, x = a cos θ, y = b sin θ are the parametric equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

where the parameter $0 \le \; < 2\pi$.

Hence the coordinates of any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

may be taken as (a cos θ, b sin θ). This point is also called the point 'θ'.

The angle θ is called the eccentric angle of the point (a cos θ, b sin θ) on the ellipse.

Equation of Chord

The equation of the chord joining the points P $\equiv$ (acos θ_1, bsin θ_1) and Q $\equiv$ (acos θ_2, bsin θ_2) is

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right).$$

POSITION OF A POINT WITH RESPECT TO AN ELLIPSE

The point $P(x_1, y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according as

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0,=0 \quad \text{or} < 0.$$

CONDITION OF TANGENCY AND POINT OF CONTACT

The condition for the line y = mx + c to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is that $c^2 = a^2m^2 + b^2$ and the coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}},\mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

Note

- x cos a + y sin a = p is a tangent if $p^2 = a^2 \cos^2 a + b^2 \sin^2 a$.
- lx + my + n = 0 is a tangent if $n^2 = a^2l^2 + b^2m^2$.

EQUATION OF TANGENT IN DIFFERENT FORMS

(i) Point Form

The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Note :

The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1.x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$, and xx by $\frac{xy_1+x_1y}{2}$. This method is used only when the equation of ellipse is a polynomial of second degree in x and y.

(ii) Parametric Form

The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (acos q, bsin q) is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1.$$

(iii) Slope Form

The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of slope 'm' is

$$y = mx \pm\sqrt{a^2m^2+b^2}$$

The coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}},\mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

Note :

- Number of Tangent Drawn From a Point

  Two tangents can be drawn from a point to an ellipse. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the ellipse.

- Director Circle

  It is the locus of points from which perpendicular tangents are drawn to the ellipse. The equation of Director Circle of the ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } x^2 + y^2 = a^2 + b^2.$$

The product of perpendicualr from the foci on any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is equal to b^2.

EQUATION OF NORMAL IN DIFFERENT FORMS

(i) Point Form

The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2.$$

(ii) Parametric Form

The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (acos q, bsin q) is

$$ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2.$$

or $$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2.$$

(iii) Slope Form

The equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of slope 'm' is

$$y=mx\pm\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$$

Notes :

- The coordinates of the points of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2+b^2m^2}},\pm\frac{mb^2}{\sqrt{a^2+b^2m^2}}\right)$$

- Condition for normality The line = mx + c is normal to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ if } c^2=\frac{m^2(a^2-b^2)^2}{(a^2+b^2m^2)}$$

- Number of Normals

In general, four normals can be drawn to an ellipse from a point in its plane i.e., there are four points on the ellipse, the normals at which it will pass through a given point. These four points are called the co-normal points.

- if $\alpha,\beta,\gamma,\delta$ are the eccentric angles of the four points on the ellipse such that the normals at these points are concurrent, then $(\alpha+\beta+\gamma+\delta)$ is an odd multiple of π.

- If α,β,γ are the eccentric angles of three points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the normals at which are concurrent, then

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0.$$

EQUATION OF THE PAIR OF TANGENTS

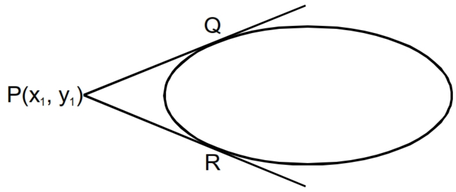
The equation of the pair of tangents drawn from a point

$P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$SS_1 = T^2$$

where $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1,\ S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

and $T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

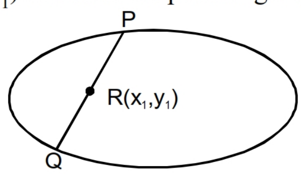


CHORD WITH A GIVEN MID POINT

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $P(x_1, y_1)$ as its middle point is given by

$$T = S_1$$

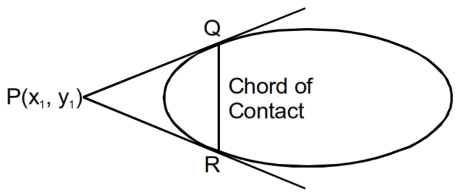
where $T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$ and $S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$.



CHORD OF CONTACT

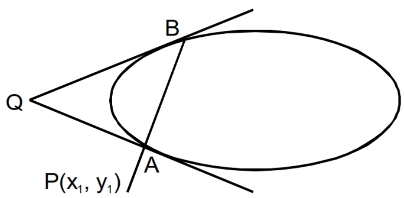
The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $T = 0$, where

$$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1.$$



POLE AND POLAR

Let P be a given point. Let a line through P intersect the ellipse at two points A and B. Let the tangents at A and B intersect at Q. The locus of point Q is a straight line called the polar of point P w.r.t. the ellipse and the point P is called the pole of the polar.



Equation of polar of a Point

The polar of a point $P(x_1, y_1)$ w.r.t. the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $T = 0$, where $T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$.

Notes :

- Polar of the focus is the directrix.

- Any tangent is the polar of its points of contact.

- Pole of a given line $lx + my + n = 0$ w.r.t. the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \left(\frac{-a^2 l}{n}.\frac{-b^2 m}{n}\right)$$

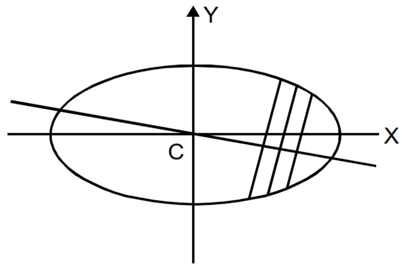
- If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$ then the polar of Q will pass through P and such points are said to be conjugate points.

- If the pole of a line $lx + my + n = 0$ lies on the another line $l'x + m'y + n' = 0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

- The point of intersection of any two lines is the pole of the line joining the pole of the two lines.

DIAMETER OF AN ELLIPSE

The locus of the middle points of a system of a parallel chords of an ellipse is called a diameter of the ellipse.



The equation of the diameter bisecting chords of slope m of the ellipse

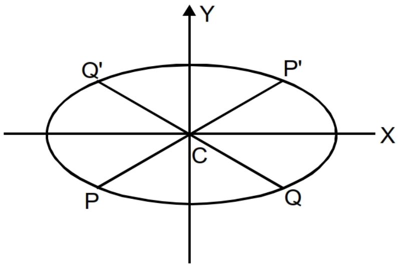
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is $$y = \frac{b^2}{a^2 m}x.$$

Note : Diameter of an ellipse always passes through its centre. Thus a diameter of an ellipse is its chord passing through its centre.

CONJUGATE DIAMETERS

Two diameters of an ellipse are said to be conjugate diameters if each bisects the chord parallel to the other.



Note :

- Major and minor axes of an ellipse is also a pair of conjugate diameters.
- If m_1 and m_2 be the slopes of the conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $m_1m_2 = \frac{-b^2}{a^2}$.
- The eccentric angles of the ends of a pair of conjugate diameters differ by a right angle.

  i.e., if PCP' and QCQ' is a pair of conjugate diameters and if eccentric angle of P is θ, then eccentric angles of Q, P', Q' (proceeding in anticlockwise direction) will be $\theta + \frac{\pi}{2}$, $\theta + \pi$ and $\theta + \frac{3\pi}{2}$ respectively.

Ans.(4)